SYDNEY BOYS HIGH
MOORE PARK, SURRY HILLS

2012
Year 11 MATHEMATICS
HSC ASSESSMENT 1

Mathematics

General Instructions:

- Reading time – 5 minutes
- Working time – 90 minutes
- Write using black or blue pen
  Black pen is preferred
- Board approved calculators may be used
- All necessary working should be shown in every question if full marks are to be awarded
- Marks may NOT be awarded for messy or badly arranged work
- Answer in simplest exact form unless otherwise stated

Total marks – 70 Marks

Section I Pages 2–4
10 marks

- Attempt Questions 1–10
- Answer on the Multiple Choice answer sheet provided.
- Allow about 15 minutes for this section

Section II Pages 5–8
60 marks

- Attempt Questions 11–14
- Allow about 75 minutes for this section
- For Questions 11–14, start a new answer booklet per question

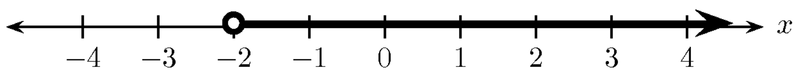
Examiner: Mr R. Elliott

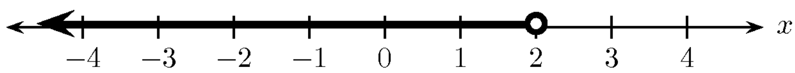
Section I — 10 marks

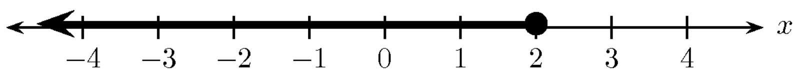
Select the alternative A, B, C, or D that best answers the question.
Fill in the response oval on your multiple choice answer sheet.

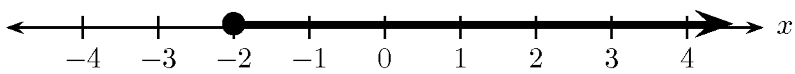
Marks

1. Which of the following graphs represents the solution of $-4x < 8$? [1]

(A) 

(B) 

(C) 

(D) 

2. What is the value of $\displaystyle\sum_{r=1}^{40}(3r-7)$? [1]

   (A) 109

   (B) 2180

   (C) 2260

   (D) 2380

3. What is the value of k if the sum of the roots of $x^2-(k-1)x+2k=0$ is equal to the product of the roots? [1]

   (A) -3

   (B) -2

   (C) -1

   (D) 1

4. What is the value of $f'(3)$ if $f(x)=3x-x^3$? [1]

   (A) $f'(3)=-24$

   (B) $f'(3)=-18$

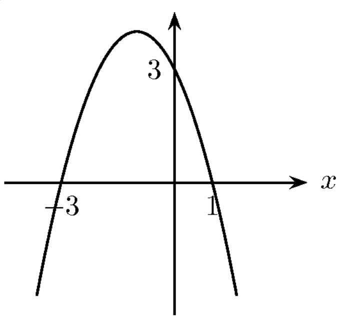
   (C) $f'(3)=0$

   (D) $f'(3)=9$

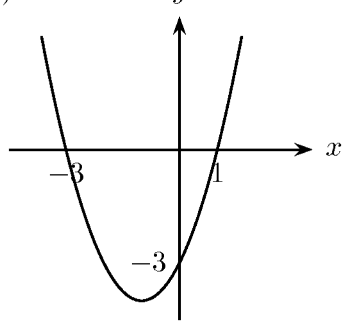
Marks

5. Which graph best represents $y = x^2 + 2x - 3$? [1]

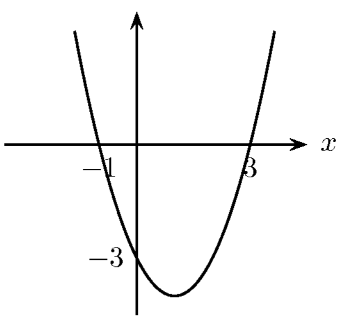
(A)



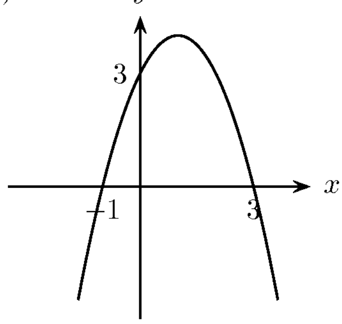
(B)



(C)



(D)



6. The equation whose roots are $4 + \sqrt{3}$ and $4 - \sqrt{3}$ is [1]

(A) $x^2 + 8x + 13$

(B) $x^2 - 8x + 13$

(C) $x^2 + 8x - 13$

(D) $x^2 - 8x - 13$

7. An infinite geometric series has a first term of 8 and a limiting sum of 12. What is the common ratio? [1]

(A) $\frac{1}{6}$

(B) $\frac{1}{4}$

(C) $\frac{1}{3}$

(D) $\frac{1}{2}$

Marks

8. Boxes are stacked in layers, where each layer contains one box fewer than the layer below. There are six boxes in the top layer, seven boxes in the next layer, and so on. There are n layers altogether. Which of the following is the correct expression for the number of boxes in the bottom layer? 1

   (A) $n + 5$

   (B) $n + 6$

   (C) $6n - 1$

   (D) $6n - 5$

9. Let $f(x) = x^2$ where x is real. If $f(k) = f(k + 1)$, then $k =$ 1

   (A) 1

   (B) $\frac{1}{2}$

   (C) $-\frac{1}{2}$

   (D) No such k exists.

10. Which of the following is/are equal to $\frac{\log_{10} a}{\log_{10} b}$? 1

    (A) $\frac{a}{b}$

    (B) $\log_{10} a - \log_{10} b$

    (C) $\log_{10}\left(\frac{a}{b}\right)$

    (D) $\log_b a$

End Multiple Choice Questions

Section II— 60 marks

Marks

Question 11 (15 marks) (use a separate answer booklet)

(a) Find the 10th term of each of these series:

(i) $5 + 2 - 1 \ldots$ [2]

(ii) $80 - 40 + 20 \ldots$ [2]

(b) A man saves \$74 from his first week's wages. He vows to increase his saving by \$2 every week he works. How much will he save in his first year? [3]

(c) The third term in a geometric series is 54 and the sixth term is 2.

(i) Find the first term and the common ratio. [2]

(ii) Find also the limiting sum. [2]

(d) Sketch the graph defined by [4]

$$f(x) = \begin{cases} 2 & \text{for} \quad x \leqslant 1 \\ x+1 & \text{for} \quad 1 < x < 3 \\ 13 - x^2 & \text{for} \quad 3 \leqslant x \leqslant 5 \end{cases}$$

Explain why this function is continuous.

Marks

Question 12 (15 marks) (use a separate answer booklet)

(a) Find the sum of 10 terms of the series **3**

$$\log_3 5 + \log_3 10 + \log_3 20 \ldots$$

Express your answer as an unsimplified logarithm.

(b) Write the formula $2y = x^2 + 6x + 12$ in the form $(x-h)^2 = 4a(y-k)$. **6**
Hence

(i) state the vertex of the parabola,

(ii) state the focal length,

(iii) write the equation of the directrix.

(c) Given $\log_3 a = 2$ and $\log_3 b = -\frac{1}{2}$:

(i) Find a and b. **2**

(ii) Evaluate $\dfrac{\sqrt{a}}{b^4}$. **2**

(d) Differentiate $y = x - x^2$ from first principles. **2**

Marks

Question 13 (15 marks) (use a separate answer booklet)

(a) For the parabola $y = 2x^2 - x - 3$:

(i) Find where it cuts the x-axis. 2

(ii) Find the values of x for which $2x^2 - x \geqslant 3$. 2

(b) Differentiate the following:

(i) $2x^3 + \dfrac{1}{\sqrt{x}} + 7$, 2

(ii) $\dfrac{3x}{x-1}$, 2

(iii) $\left(5x^2 - 1\right)^3$, 2

(iv) $3x(2-x)^4$. 2

(c) What is the domain and range of $y = \dfrac{|x|}{x} - x$? 3
Sketch the function on the number-plane.

Marks

Question 14 (15 marks) (use a separate answer booklet)

(a) (i) Find $\log_4 7$ correct to 3 decimal places. 2

(ii) Simplify $\log_a \sqrt{x} + 2\log_a x - \log_a \frac{1}{x}$. 2

(iii) Solve for x, 2

$$\log_2 8 - 2\log_4 x = 0.$$

(b) For the quadratic $x^2 + bx + 5 = 0$, one of the roots is known to be -2.

(i) Find the other root. 2

(ii) Find b. 2

(c) (i) Find the equation of the tangent to the curve $y = \dfrac{c^2}{x}$ at the point $P(x_1, y_1)$. (c is a constant.) 2

(ii) Let the tangent cut the x- and y-axes at A and B respectively. Show that the area of $\triangle AOB$ is independent of the position of P. 3

End of Paper

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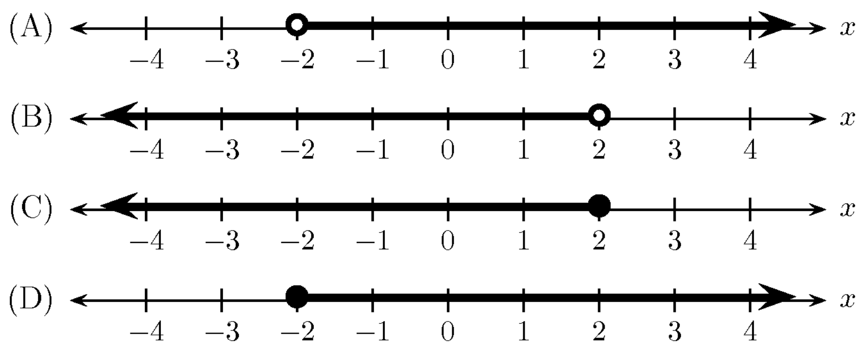
Mathematics Solutions

Section I— 10 marks

Select the alternative A, B, C, or D that best answers the question.
Fill in the response oval on your multiple choice answer sheet.

Marks

1. Which of the following graphs represents the solution of $-4x < 8$? 1



Solution: $x > -2$, $\therefore$ (A)

2. What is the value of $\displaystyle\sum_{r=1}^{40}(3r-7)$? 1

   (A) 109

   (B) 2180

   (C) 2260

   (D) 2380

Solution: $\frac{40}{2}\big([3\times 1-7]+[3\times 40-7]\big) = 2180$, $\therefore$ (B).

3. What is the value of k if the sum of the roots of $x^2-(k-1)x+2k=0$ is equal to the product of the roots? 1

   (A) -3

   (B) -2

   (C) -1

   (D) 1

Solution:
$$\begin{aligned} k-1 &= 2k, \\ -1 &= k, \text{ so (C).} \end{aligned}$$

4. What is the value of $f'(3)$ if $f(x) = 3x - x^3$? [1]

(A) $f'(3) = -24$

(B) $f'(3) = -18$

(C) $f'(3) = 0$

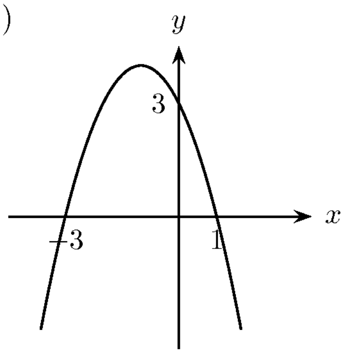
(D) $f'(3) = 9$

Solution: $f'(x) = 3 - 3x^2,$
$f'(3) = 3 - 27,$
$= -24,$ so answer (A).

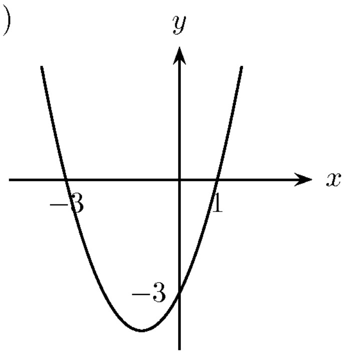
Marks

5. Which graph best represents $y = x^2 + 2x - 3$? [1]

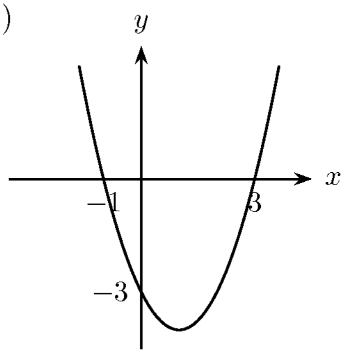
(A)



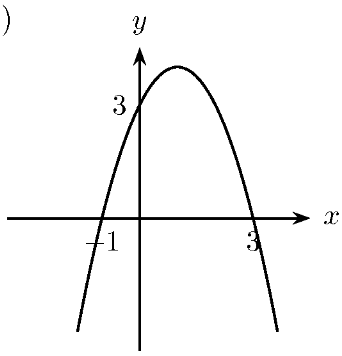
(B)



(C)



(D)



Solution: When $x = 0,\ y = -3$, so B or C.
Now $y = (x+3)(x-1)$, thus when $y = 0,\ x = -3,\ 1$—so (B).

6. The equation whose roots are $4+\sqrt{3}$ and $4-\sqrt{3}$ is 1

(A) $x^2+8x+13$

(B) $x^2-8x+13$

(C) $x^2+8x-13$

(D) $x^2-8x-13$

Solution: (B), as sum of roots is 8 and product of roots is 13.

7. An infinite geometric series has a first term of 8 and a limiting sum of 12. What is the common ratio? 1

(A) $\frac{1}{6}$

(B) $\frac{1}{4}$

(C) $\frac{1}{3}$

(D) $\frac{1}{2}$

Solution:

$$\begin{aligned} 12 &= \frac{8}{1-r} \\ 12-12r &= 8 \\ -12r &= -4 \\ r &= \frac{1}{3}, \text{ so (C).} \end{aligned}$$

Marks

8. Boxes are stacked in layers, where each layer contains one box fewer than the layer below. There are six boxes in the top layer, seven boxes in the next layer, and so on. There are n layers altogether. Which of the following is the correct expression for the number of boxes in the bottom layer? 1

(A) $n+5$

(B) $n+6$

(C) $6n-1$

(D) $6n-5$

Solution:

1	6	
2	$6+1$	$6+2-1$
3	$6+2$	$6+3-1$
$\vdots$	$\vdots$	$\vdots$
n	$n+5$	$6+n-1$, so (A).

9. Let $f(x)=x^2$ where x is real. If $f(k)=f(k+1)$, then $k=$ 1

   (A) 1

   (B) $\dfrac{1}{2}$

   (C) $-\dfrac{1}{2}$

   (D) No such k exists.

Solution: (C), as $f\left(-\frac{1}{2}\right)=\dfrac{1}{4}$ and $f\left(1-\frac{1}{2}\right)=f\left(\frac{1}{2}\right)=\dfrac{1}{4}$.

10. Which of the following is/are equal to $\dfrac{\log_{10} a}{\log_{10} b}$? 1

   (A) $\dfrac{a}{b}$

   (B) $\log_{10} a - \log_{10} b$

   (C) $\log_{10}\left(\dfrac{a}{b}\right)$

   (D) $\log_b a$

Solution: (D), using change of base law.

End Multiple Choice Questions

Section II— 60 marks

Marks

Question 11 (15 marks) (use a separate answer booklet)

(a) Find the 10th term of each of these series:

(i) $5 + 2 - 1 \ldots$ [2]

Solution: $8 - 3 \times 10 = -22$.

(ii) $80 - 40 + 20 \ldots$ [2]

Solution: $80 \times \left(-\frac{1}{2}\right)^{10-1} = -\dfrac{5}{32}$ or -0.15625.

(b) A man saves \$74 from his first week's wages. He vows to increase his saving by \$2 every week he works. How much will he save in his first year? [3]

Solution:

$$\begin{aligned}
\text{Week 1 saving} &= \$74, \\
\text{Week 2 saving} &= \$76 \ (i.e.\ 74 + 2,) \\
\text{Week 3 saving} &= \$78 \ (i.e.\ 74 + 2 \times [3 - 1]), \\
\text{Week 52 saving} &= \$(74 + 2 \times [52 - 1]), \\
&= \$176. \\
\text{Total savings} &= \frac{52}{2}(\$74 + \$176) \text{ or } \frac{52}{2}(2 \times \$74 + \$2[52 - 1]), \\
&= \$6500.
\end{aligned}$$

(c) The third term in a geometric series is 54 and the sixth term is 2.

(i) Find the first term and the common ratio. [2]

Solution:

$$\begin{aligned}
ar^2 &= 54, \\
ar^5 &= 2, \\
r^3 &= \frac{2}{54}, \\
&= \frac{1}{3^3}, \\
\therefore r &= \tfrac{1}{3}. \\
a &= 54 \times 3^2, \\
&= 486.
\end{aligned}$$

I.e. the first term is 486 and the common ratio is $\dfrac{1}{3}$.

(ii) Find also the limiting sum. **2**

Solution: If $r = \frac{1}{3}$,

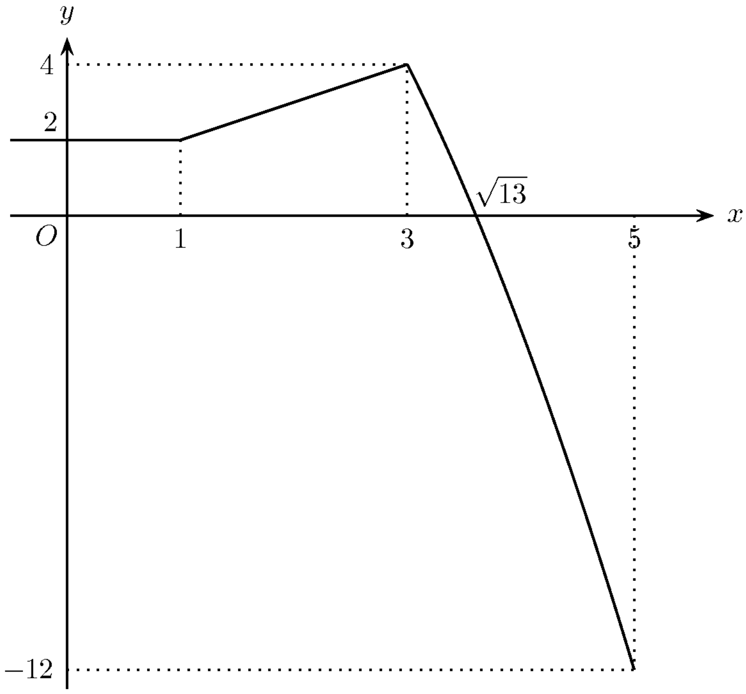
$$\begin{aligned} \text{then sum} &= \frac{486}{1 - \frac{1}{3}}, \\ &= 729. \end{aligned}$$

(d) Sketch the graph defined by **4**

$$f(x) = \begin{cases} 2 & \text{for} \quad x \leqslant 1 \\ x + 1 & \text{for} \quad 1 < x < 3 \\ 13 - x^2 & \text{for} \quad 3 \leqslant x \leqslant 5 \end{cases}$$

Explain why this function is continuous.

Solution:



As $x \longrightarrow 1^-$, $\lim f(x) \longrightarrow 2$, and as $x \longrightarrow 1^+$, $\lim f(x) \longrightarrow 2$.
As $x \longrightarrow 3^-$, $\lim f(x) \longrightarrow 4$, and as $x \longrightarrow 3^+$, $\lim f(x) \longrightarrow 4$.

Marks

Question 12 (15 marks) (use a separate answer booklet)

(a) Find the sum of 10 terms of the series **3**

$$\log_3 5 + \log_3 10 + \log_3 20 \ldots$$

Express your answer as an unsimplified logarithm.

Solution: Method 1—

$$\begin{aligned}\log_3 (5 \times 10 \times 20 \times \cdots \times \{5 \times 2^9\}) &= \log_3 \left(5^{10} \times 2^{(0+1+2+\cdots+9)}\right),\\ &= \log_3 (5^{10} \times 2^{45}).\end{aligned}$$

Solution: Method 2—

$$\begin{aligned}T_n &= a + (n-1)d.\\ d &= T_2 - T_1 = T_3 - T_2,\\ &= \log_3 10 - \log_3 5 = \log_3 20 - \log_3 10,\\ &= \log_3 \left(\frac{10}{5}\right) = \log_3 \left(\frac{20}{10}\right),\\ &= \log_3 2.\\ T_n &= \log_3 5 + (n-1)\log_3 2.\\ T_{10} &= \log_3 5 + 9\log_3 2.\\ S_n &= \frac{n}{2}(a + \ell).\\ S_{10} &= \frac{10}{2}(\log_3 5 + \log_3 5 + 9\log_3 2),\\ &= 5(2\log_3 5 + 9\log_3 2).\end{aligned}$$

(b) Write the formula $2y = x^2 + 6x + 12$ in the form $(x-h)^2 = 4a(y-k)$. **6**

Solution:

$$\begin{aligned}x^2 + 6x + 3^2 &= 2y - 12 + 9,\\ (x+3)^2 &= 2y - 3,\\ \textit{i.e. } (x+3)^2 &= 4 \times \tfrac{1}{2}\left(y - \tfrac{3}{2}\right).\end{aligned}$$

Hence (i) state the vertex of the parabola,

Solution: $\big(-3,\, 1\tfrac{1}{2}\big)$

(ii) state the focal length,

Solution: $\frac{1}{2}$

(iii) write the equation of the directrix.

Solution: $y = 1$

(c) Given $\log_3 a = 2$ and $\log_3 b = -\frac{1}{2}$:

(i) Find a and b. 2

Solution: $$\begin{aligned} a &= 3^2, & b &= 3^{-1/2}, \\ &= 9. & &= \frac{1}{\sqrt{3}}. \end{aligned}$$

(ii) Evaluate $\dfrac{\sqrt{a}}{b^4}$. 2

Solution: $$\begin{aligned} \sqrt{9} \div \left(\frac{1}{\sqrt{3}}\right)^4 &= 3 \times 9, \\ &= 27. \end{aligned}$$

(d) Differentiate $y = x - x^2$ from first principles. 2

Solution: $$\begin{aligned} y' &= \lim_{h \longrightarrow 0} \left\{ \frac{\big((x+h) - (x+h)^2\big) - (x - x^2)}{h} \right\}, \\ &= \lim_{h \longrightarrow 0} \left\{ \frac{x + h - x^2 - 2xh - h^2 - x + x^2}{h} \right\}, \\ &= \lim_{h \longrightarrow 0} \left\{ \frac{h - 2xh - h^2}{h} \right\}, \\ &= \lim_{h \longrightarrow 0} \{1 - 2x - h\}, \\ &= 1 - 2x. \end{aligned}$$

Marks

Question 13 (15 marks) (use a separate answer booklet)

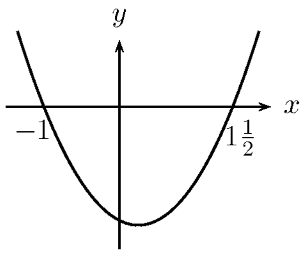
(a) For the parabola $y = 2x^2 - x - 3$:

(i) Find where it cuts the x-axis. **2**

Solution:
$$\begin{aligned} 0 &= 2x^2 - x - 3, \\ &= (2x-3)(x+1), \\ \therefore x &= -1 \text{ or } \tfrac{3}{2}. \end{aligned}$$
So the parabola cuts the x-axis at -1 and $1\frac{1}{2}$.

(ii) Find the values of x for which $2x^2 - x \geqslant 3$. **2**

Solution:



$\therefore x \leqslant -1$ and $x \geqslant 1\frac{1}{2}$.

(b) Differentiate the following:

(i) $2x^3 + \dfrac{1}{\sqrt{x}} + 7$, **2**

Solution:
$$\begin{aligned} \frac{d}{dx}\left(2x^3 + x^{-\frac{1}{2}} + 7\right) &= 3 \times 2x^2 - \tfrac{1}{2} \times x^{-1\frac{1}{2}} + 0, \\ &= 6x^2 - \frac{1}{2x\sqrt{x}}. \end{aligned}$$

(ii) $\dfrac{3x}{x-1}$, **2**

Solution:
$$\begin{aligned} \frac{d}{dx}\left(\frac{3x}{x-1}\right) &= \frac{(x-1) \times 3 - 3x \times 1}{(x-1)^2}, \\ &= \frac{-3}{(x-1)^2}. \end{aligned}$$

(iii) $\left(5x^2 - 1\right)^3$, 2

Solution: Put $y = u^3$, where $u = 5x^2 - 1$,

$$\frac{dy}{du} = 3u^2, \quad \text{and } \frac{du}{dx} = 10x.$$

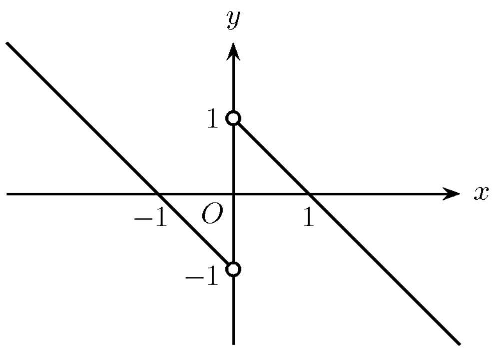
$$\text{Now } \frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx},$$
$$= 3(5x^2 - 1)^2 \times 10x,$$
$$= 30x(5x^2 - 1)^2.$$

(iv) $3x(2 - x)^4$. 2

Solution:
$$\frac{d}{dx}\left(3x(2-x)^4\right) = 3(2-x)^4 + 3x \times 4 \times (-1)(2-x)^3,$$
$$= 3(2-x)^3\left((2-x) - 4x\right),$$
$$= 3(2-5x)(2-x)^3.$$

(c) What is the domain and range of $y = \dfrac{|x|}{x} - x$? 3
Sketch the function on the number-plane.

Solution: Domain: all real x but $x \neq 0$.
Range: all real y.



Marks

Question 14 (15 marks) (use a separate answer booklet)

(a) (i) Find $\log_4 7$ correct to 3 decimal places. 2

Solution: $\dfrac{\log_{10} 7}{\log_{10} 4}$ or $\dfrac{\ln 7}{\ln 4} \approx 1.404$ (3 dec. pl.).

(ii) Simplify $\log_a \sqrt{x} + 2\log_a x - \log_a \frac{1}{x}$. 2

Solution: Method 1—

$$\begin{aligned} \log_a x^{\frac{1}{2}} + \log_a x^2 - \log_a x^{-1} &= \log_a x^{\frac{1}{2}+2-(-1)}, \\ &= \log_a x^{\frac{7}{2}} \text{ or } \frac{7}{2}\log_a x. \end{aligned}$$

Solution: Method 2—

$$\tfrac{1}{2}\log_a x + 2\log_a x - (-\log_a x) = \frac{7}{2}\log_a x \text{ or } \log_a x^{\frac{7}{2}}.$$

(iii) Solve for x, 2

$$\log_2 8 - 2\log_4 x = 0.$$

Solution:

$$\begin{aligned} \log_2 2^3 &= 2\log_4 x, \\ \log_4 x &= \frac{3}{2}, \\ x &= 4^{\frac{3}{2}}, \\ &= 8. \end{aligned}$$

(b) For the quadratic $x^2 + bx + 5 = 0$, one of the roots is known to be -2.

(i) Find the other root. 2

Solution: Let the other root be α.

Product of roots, $-2\alpha = 5$,

$$\alpha = -\frac{5}{2}.$$

i.e. The other root is $-2\frac{1}{2}$.

(ii) Find b. 2

Solution: Sum of roots, $-2 - 2\frac{1}{2} = -b$,

$$b = 4\tfrac{1}{2}.$$

(c) (i) Find the equation of the tangent to the curve $y = \dfrac{c^2}{x}$ at the point $P(x_1, y_1)$. (c is a constant.) **2**

> **Solution:** $\dfrac{dy}{dx} = -\dfrac{c^2}{x^2}.$
>
> $$\begin{aligned} \text{Tangent: } y - y_1 &= \frac{-c^2}{x_1^2}(x - x_1), \\ x_1^2 y - x_1^2 y_1 &= x_1 c^2 - x c^2. \end{aligned}$$

(ii) Let the tangent cut the x- and y-axes at A and B respectively. Show that the area of $\triangle AOB$ is independent of the position of P. **3**

> **Solution:**
>
> $$\begin{aligned} \text{At } A,\ y &= 0, \\ -x_1^2 y_1 &= x_1 c^2 - x c^2, \\ x &= \frac{x_1(c^2 + x_1 y_1)}{c^2}. \\ \text{At } B,\ x &= 0, \\ x_1^2 y - x_1^2 y_1 &= x_1 c^2, \\ y &= \frac{x_1 y_1 + c^2}{x_1}. \\ \text{Area } \triangle AOB &= \frac{1}{2}.\frac{x_1(c^2 + x_1 y_1)}{c^2}.\frac{(x_1 y_1 + c^2)}{x_1}, \\ &= \frac{(x_1 y_1 + c^2)^2}{2c^2}. \\ \text{But } x_1 y_1 &= c^2 \text{ (from the equation of the curve)}, \\ \therefore \text{ area } \triangle AOB &= \frac{(2c^2)^2}{2c^2}, \\ &= 2c^2 \text{ which is independent of } x_1 \text{ and } y_1. \end{aligned}$$

End of Paper